

Q CURVATURE ON A CLASS OF MANIFOLDS WITH DIMENSION AT LEAST 5

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ABSTRACT. For a smooth compact Riemannian manifold with positive Yamabe invariant, positive Q curvature and dimension at least 5, we prove the existence of a conformal metric with constant Q curvature. Our approach is based on the study of extremal problem for a new functional involving the Paneitz operator.

1. INTRODUCTION

Recall the definition of the 4th order Paneitz operator and its associated Q curvature [B, P]: when (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 3$, the Q curvature is given by

$$\begin{aligned} Q &= -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2}|Rc|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}R^2 \\ &= -\Delta J - 2|A|^2 + \frac{n}{2}J^2. \end{aligned} \quad (1.1)$$

Here R is the scalar curvature, Rc is the Ricci tensor and

$$J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2}(Rc - Jg). \quad (1.2)$$

The Paneitz operator is given by

$$\begin{aligned} P\varphi &= \Delta^2\varphi + \frac{4}{n-2}\operatorname{div}(Rc(\nabla\varphi, e_i)e_i) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)}\operatorname{div}(R\nabla\varphi) + \frac{n-4}{2}Q\varphi \\ &= \Delta^2\varphi + \operatorname{div}(4A(\nabla\varphi, e_i)e_i - (n-2)J\nabla\varphi) + \frac{n-4}{2}Q\varphi. \end{aligned} \quad (1.3)$$

Here e_1, \dots, e_n is a local orthonormal frame with respect to g . When $n \neq 4$, under a conformal change of the metric, the operator satisfies

$$P_{\rho^{\frac{4}{n-4}}g}\varphi = \rho^{-\frac{n+4}{n-4}}P_g(\rho\varphi). \quad (1.4)$$

This is similar to the conformal Laplacian operator, which appears naturally when considering transformation law of the scalar curvature under conformal change of metric in dimension greater than 2 ([LP]). As a consequence we have

$$P_{\rho^{\frac{4}{n-4}}g}\varphi \cdot \psi d\mu_{\rho^{\frac{4}{n-4}}g} = P_g(\rho\varphi) \cdot \rho\psi d\mu_g. \quad (1.5)$$

Here μ_g is the measure associated with metric g .

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In dimension 4, the Paneitz operator is given by

$$P\varphi = \Delta^2\varphi + 2 \operatorname{div} (Rc(\nabla\varphi, e_i) e_i) - \frac{2}{3} \operatorname{div} (R\nabla\varphi), \quad (1.6)$$

and its conformal covariance property takes the form

$$P_{e^{2w}g}\varphi = e^{-4w}P_g\varphi. \quad (1.7)$$

Following the basic work [CGY] in dimension 4 on the 4th order Q curvature equation, there has been several studies on this equation in dimension 3 by [HY1, XY2, YZ], and in dimensions greater than 4 by [DHL, HeR1, HeR2, HuR, QR1, QR2].

While it is important to determine conditions under which the Paneitz operator is positive, we discover that it is sufficient for our purpose in this article to determine when its Green's function is positive. This is a property that is conformally invariant: observe that by (1.4),

$$\ker P_g = 0 \Leftrightarrow \ker P_{\rho^{\frac{4}{n-4}}g} = 0, \quad (1.8)$$

and under this assumption, the Green's functions G_P satisfy the transformation law

$$G_{P, \rho^{\frac{4}{n-4}}g}(p, q) = \rho(p)^{-1} \rho(q)^{-1} G_{P,g}(p, q). \quad (1.9)$$

In analogy with the preliminary study of the classical Yamabe problem ([LP]), the first question would be whether one can find a conformal invariant condition for the existence of a conformal metric with positive Q curvature. In the case Yamabe invariant $Y(g)$ is positive, the existence of a conformal metric with positive Q curvature is equivalent to the requirements that $\ker P = 0$ and the Green's function $G_P > 0$ ([HY4]).

The basic question of interest is to find constant Q curvature metric in a conformal class, in the same spirit as Yamabe problem. The main aim of the present article is to prove the following

Theorem 1.1. *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then $\ker P = 0$, the Green's function of P is positive and there exists a conformal metric \tilde{g} with $\tilde{Q} = 1$.*

Remark 1.1. *Let (M^n, g) be a smooth compact Riemannian manifold with $n \geq 5$, $Y(g) > 0$. Denote $L = -\frac{4(n-1)}{n-2}\Delta + R$ as the conformal Laplacian operator and for $p \in M$, $G_{L,p}$ as the Green's function of L with pole at p . Define*

$$\Gamma_1(p, q) = 2^{\frac{n-6}{n-2}} n^{-\frac{2}{n-2}} (n-1)^{\frac{n-4}{n-2}} (n-2)^{-3} \omega_n^{-\frac{2}{n-2}} G_L(p, q)^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}}g} \right|_g^2(q).$$

Here ω_n is the volume of unit ball in \mathbb{R}^n , $G_L(p, q) = G_{L,p}(q)$. The associated integral operator T_{Γ_1} is given by

$$T_{\Gamma_1}(\varphi)(p) = \int_M \Gamma_1(p, q) \varphi(q) d\mu(q)$$

for any nice function φ on M . In [HY5], it is shown that the spectrum $\sigma(T_{\Gamma_1})$ and spectral radius $r_\sigma(T_{\Gamma_1})$ are conformal invariants, moreover the following statements are equivalent:

- (1) there exists a conformal metric \tilde{g} with $\tilde{Q} > 0$.

- (2) $\ker P = 0$ and the Green's function of Paneitz operator $G_P(p, q) > 0$ for $p \neq q$.
- (3) $\ker P = 0$ and there exists $p \in M$ such that $G_P(p, q) > 0$ for $q \neq p$.
- (4) $r_\sigma(T_{\Gamma_1}) < 1$.

Under the assumption $Q \geq 0$ and not identically zero, we have $r_\sigma(T_{\Gamma_1}) < 1$.

The fundamental difficulty of the lack of maximum principle in this 4th order equation has recently been overcome by the work in [GM]. Following this development, similar results in dimension 3 were proved in [HY3, HY4] (see also closely related [HY2]). Dimension 4 case does not suffer from this difficulty and was treated in many articles like [CY, DM, FR] and so on. For a locally conformally flat manifold with positive Yamabe invariant and Poincare exponent less than $\frac{n-4}{2}$ (see [SY]), Theorem 1.1 was proved in [QR2] by apriori estimates and connecting the equation to Yamabe equation through a path of integral equations. Under the slightly more stringent conditions $R > 0$ and $Q > 0$, Theorem 1.1 was proved in [GM] through the study of a non-local flow. Here we will derive Theorem 1.1 by maximizing a functional (see (1.16) and (2.2)) involving the Paneitz operator (see Theorem 1.3 for more details).

For $u, v \in C^\infty(M)$, we denote the quadratic form associated to P as

$$\begin{aligned}
 & E(u, v) \\
 &= \int_M Pu \cdot v d\mu \\
 &= \int_M \left(\Delta u \Delta v - \frac{4}{n-2} R c(\nabla u, \nabla v) + \frac{n^2 - 4n + 8}{2(n-1)(n-2)} R \nabla u \cdot \nabla v \right. \\
 &\quad \left. + \frac{n-4}{2} Q uv \right) d\mu \\
 &= \int_M \left(\Delta u \Delta v - 4A(\nabla u, \nabla v) + (n-2)J \nabla u \cdot \nabla v + \frac{n-4}{2} Q uv \right) d\mu,
 \end{aligned} \tag{1.10}$$

and

$$E(u) = E(u, u). \tag{1.11}$$

By the integration by parts formula in (1.10) we know that $E(u, v)$ extends continuously to $u, v \in H^2(M)$.

To find the metric \tilde{g} in Theorem 1.1, we write $\tilde{g} = \rho^{\frac{4}{n-4}} g$, then the equation $\tilde{Q} = 1$ becomes

$$P_g \rho = \frac{n-4}{2} \rho^{\frac{n+4}{n-4}}, \quad \rho \in C^\infty(M), \rho > 0. \tag{1.12}$$

Let

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|_{L^{\frac{2n}{n-4}}}^2}, \tag{1.13}$$

then $Y_4\left(\tau^{\frac{4}{n-4}} g\right) = Y_4(g)$ for any positive smooth function τ . Hence $Y_4(g)$ is a conformal invariant. If (M, g) is not locally conformally flat and $n \geq 8$, or (M, g) is locally conformally flat with $Y(g) > 0$, $\ker P = 0$ and the Green's function of P , $G_P > 0$, or $n = 5, 6, 7$ with $Y(g) > 0$, $\ker P = 0$ and $G_P > 0$, one can show $Y_4(g)$ is achieved (see [ER, GM, R]), but in general it is difficult to know whether the minimizer is positive. Under the additional assumption $Y_4(g) > 0$ and $G_P > 0$, it

was observed in [R] that the minimizer cannot change sign. Combining this with the positivity criterion of Green's function in [HY4], we arrive at

Theorem 1.2. *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Y_4(g) > 0$, $Q \geq 0$ and not identically zero, then*

- (1) $Y_4(g) \leq Y_4(S^n)$, and equality holds if and only if (M, g) is conformally diffeomorphic to the standard sphere.
- (2) $Y_4(g)$ is always achieved. Any minimizer must be smooth and cannot change sign. In particular we can find a constant Q curvature metric in the conformal class.
- (3) If (M, g) is not conformally diffeomorphic to the standard sphere, then the set of all minimizers u for $Y_4(g)$, after normalizing with $\|u\|_{L^{\frac{2n}{n-4}}} = 1$, is compact in C^∞ topology.

Note the positivity of $Y_4(g)$ is equivalent to the positivity of Paneitz operator P . There are several criterion for the positivity of P (see [CHY, Theorem 1.6] and [GM, XY1]). On the other hand, in a recent preprint [GHL], it is proved that if (M, g) is a smooth compact Riemannian manifold with dimension $n \geq 6$, and $Y(g) > 0$, $Y_4(g) > 0$, then we can find a conformal metric \tilde{g} with $\tilde{R} > 0$ and $\tilde{Q} > 0$. In particular, it follows from [GM] that any conformal metric with constant Q curvature must have positive scalar curvature. Similar statement for $n = 5$ is likely to be true but could not be justified due to the approach there.

In general it is not known whether $Y(g) > 0$, $Q \geq 0$ and not identically zero would imply $Y_4(g) > 0$. To get around this difficulty when proving Theorem 1.1 we note that by [HY4, Proposition 1.1] if $Y(g) > 0$, $Q \geq 0$ and not identically zero then $\ker P = 0$, and the Green's function of P , $G_P > 0$. Hence we can define an integral operator (the inverse of P) as

$$G_P f(p) = \int_M G_P(p, q) f(q) d\mu(q). \quad (1.14)$$

If we denote $f = \rho^{\frac{n+4}{n-4}}$, then equation (1.12) becomes

$$G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}, \quad f \in C^\infty(M), f > 0. \quad (1.15)$$

Let

$$\begin{aligned} \Theta_4(g) &= \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2} \\ &= \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M \times M} G_P(p, q) f(p) f(q) d\mu(p) d\mu(q)}{\|f\|_{L^{\frac{2n}{n+4}}}^2}. \end{aligned} \quad (1.16)$$

It follows from the classical Hardy-Littlewood-Sobolev inequality ([St]) that $\Theta_4(g)$ is always finite. Moreover it follows from (1.9) that for a positive smooth function ρ , $\Theta_4\left(\rho^{\frac{4}{n-4}}g\right) = \Theta_4(g)$ i.e. $\Theta_4(g)$ is a conformal invariant. If $\Theta_4(g)$ is achieved by a maximizer f , using the fact that $G_P > 0$, we easily deduce that f cannot change

sign. $\Theta_4(g)$ has a nice geometric description (see Lemma 2.1):

$$\Theta_4(g) = \frac{2}{n-4} \sup \left\{ \frac{\int_M \tilde{Q} d\tilde{\mu}}{\|\tilde{Q}\|_{L^{\frac{2n}{n+4}}(M, d\tilde{\mu})}^2} : \tilde{g} \in [g] \right\} \quad (1.17)$$

Here $[g]$ denotes the conformal class of g i.e.

$$[g] = \{ \rho^2 g : \rho \in C^\infty(M), \rho > 0 \}. \quad (1.18)$$

Theorem 1.3. *Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then*

- (1) $\Theta_4(g) \geq \Theta_4(S^n)$, here S^n has the standard metric. $\Theta_4(g) = \Theta_4(S^n)$ if and only if (M, g) is conformally diffeomorphic to the standard sphere.
- (2) $\Theta_4(g)$ is always achieved. Any maximizer f must be smooth and cannot change sign. If $f > 0$, then after scaling we have $G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}$ i.e. $Q_{f^{\frac{4}{n+4}}g} = 1$.
- (3) If (M, g) is not conformally diffeomorphic to the standard sphere, then the set of all maximizers f for $\Theta_4(g)$, after normalizing with $\|f\|_{L^{\frac{2n}{n+4}}} = 1$, is compact in the C^∞ topology.

It is worthwhile to note the similarity of Theorem 1.2 and 1.3 to classical Yamabe problem ([LP, S]) and the integral equation considered in [HWY1, HWY2]. Indeed, the formulation of our approach follows that of [HWY2]. Integral equation formulation of the Q curvature equation was used in [QR2]. A similar functional for the conformal Laplacian operator, Θ_2 (see (4.8)) is also considered in [DoZ]. In Section 2 below we will first give other expressions for $\Theta_4(g)$ and discuss its relation with $Y_4(g)$, then we will derive the concentration compactness principle for the extremal problem of $\Theta_4(g)$ and find the asymptotic expansion formula for the Green's function of Paneitz operator. In Section 3 we will show that maximizers always exist and that they are smooth. In particular Theorem 1.3 will follow. At last, in Section 4 we will prove Theorem 1.2. Moreover we will show the approach to Theorem 1.3 gives another way to find constant scalar curvature metrics in a conformal class.

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2. SOME PREPARATIONS

2.1. The conformal invariants $Y_4(g)$, $Y_4^+(g)$ and $\Theta_4(g)$. Throughout this subsection we will assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$. Recall that

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|_{L^{\frac{2n}{n-4}}}^2} = \inf_{u \in C^\infty(M) \setminus \{0\}} \frac{\int_M Pu \cdot u d\mu}{\|u\|_{L^{\frac{2n}{n-4}}}^2}. \quad (2.1)$$

If in addition $Y(g) > 0$, $Q \geq 0$ and not identically zero, then

$$\begin{aligned}\Theta_4(g) &= \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2} \\ &= \sup_{u \in W^{4, \frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2}.\end{aligned}\tag{2.2}$$

The second equality in (2.2) will be very useful for us later on because the expression is local. It will facilitate our calculations in estimating $\Theta_4(g)$. $\Theta_4(g)$ also has a geometric description.

Lemma 2.1. *If $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then*

$$\Theta_4(g) = \frac{2}{n-4} \sup \left\{ \frac{\int_M \tilde{Q} d\tilde{\mu}}{\|\tilde{Q}\|_{L^{\frac{2n}{n+4}}(M, d\tilde{\mu})}^2} : \tilde{g} \in [g] \right\}.\tag{2.3}$$

Proof. Note that

$$\begin{aligned}& \frac{2}{n-4} \sup \left\{ \frac{\int_M \tilde{Q} d\tilde{\mu}}{\|\tilde{Q}\|_{L^{\frac{2n}{n+4}}(M, d\tilde{\mu})}^2} : \tilde{g} \in [g] \right\} \\ &= \sup \left\{ \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} : u \in C^\infty(M), u > 0 \right\} \\ &\leq \Theta_4(g).\end{aligned}$$

On the other hand, by the positivity of G_P we have

$$\begin{aligned}\Theta_4(g) &= \sup \left\{ \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2} : f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}, f \geq 0 \right\} \\ &= \sup \left\{ \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2} : f \in C^\infty(M) \setminus \{0\}, f \geq 0 \right\} \\ &= \sup \left\{ \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} : u \in C^\infty(M) \setminus \{0\}, Pu \geq 0 \right\} \\ &\leq \sup \left\{ \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} : u \in C^\infty(M), u > 0 \right\} \\ &= \frac{2}{n-4} \sup \left\{ \frac{\int_M \tilde{Q} d\tilde{\mu}}{\|\tilde{Q}\|_{L^{\frac{2n}{n+4}}(M, d\tilde{\mu})}^2} : \tilde{g} \in [g] \right\}.\end{aligned}$$

In between we have used the fact for smooth function u , $Pu \geq 0$ and u not identically zero implies $u > 0$. ■

To better understand the relation between $Y_4(g)$ and $\Theta_4(g)$, we define

$$\begin{aligned} Y_4^+(g) &= \inf \left\{ \frac{\int_M Pu \cdot u d\mu}{\|u\|_{L^{\frac{2n}{n-4}}}^2} : u \in C^\infty(M), u > 0 \right\} \\ &= \frac{n-4}{2} \inf \left\{ \frac{\int_M \tilde{Q} d\tilde{\mu}}{(\tilde{\mu}(M))^{\frac{n-4}{n}}} : \tilde{g} \in [g] \right\}. \end{aligned} \quad (2.4)$$

Clearly we have

$$Y_4(g) \leq Y_4^+(g). \quad (2.5)$$

Lemma 2.2. *If $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then*

$$Y_4^+(g) \Theta_4(g) \leq 1. \quad (2.6)$$

Moreover if $Y_4^+(g)$ is achieved, then $Y_4^+(g) \Theta_4(g) = 1$ and $\Theta_4(g)$ must be achieved too.

Proof. It is clear that $\Theta_4(g) > 0$. To prove the inequality we only need to deal with the case $Y_4^+(g) > 0$. Under this assumption for $u \in C^\infty(M)$, $u > 0$, we have $\int_M Pu \cdot u d\mu > 0$. By Holder's inequality we have

$$\frac{(\int_M Pu \cdot u d\mu)^2}{\|u\|_{L^{\frac{2n}{n-4}}}^2 \|Pu\|_{L^{\frac{2n}{n+4}}}^2} \leq 1.$$

It follows that

$$Y_4^+(g) \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} \leq 1.$$

By the proof of Lemma 2.1 we have

$$\Theta_4(g) = \sup \left\{ \frac{\int_M Pv \cdot v d\mu}{\|Pv\|_{L^{\frac{2n}{n+4}}}^2} : v \in C^\infty(M), v > 0 \right\},$$

hence $Y_4^+(g) \Theta_4(g) \leq 1$.

If $Y_4^+(g)$ is achieved, say at $u \in C^\infty(M)$, $u > 0$, then

$$Pu = \kappa u^{\frac{n+4}{n-4}}$$

for some constant κ . Since $G_P > 0$, we see that $\kappa > 0$. Hence

$$\Theta_4(g) \geq \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} = \frac{1}{\kappa} \|u\|_{L^{\frac{2n}{n-4}}}^{-\frac{8}{n-4}} = \frac{1}{Y_4^+(g)} \geq \Theta_4(g).$$

Hence all the inequalities are equalities. $\Theta_4(g) = \frac{1}{Y_4^+(g)}$ and it is achieved at u too. ■

Remark 2.1. *Assume $Y_4^+(g) \Theta_4(g) = 1$. Later we will show that $\Theta_4(g)$ is always achieved by positive smooth functions i.e.*

$$\Theta_4(g) = \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n-4}}}^2} = \frac{\int_M Pv \cdot v d\mu}{\|Pv\|_{L^{\frac{2n}{n+4}}}^2},$$

here $f \in C^\infty(M)$, $f > 0$, $v = G_P f$. Hence $v \in C^\infty(M)$, $v > 0$ and

$$Pv = \kappa v^{\frac{n+4}{n-4}}$$

for some constant κ . Using $G_P > 0$ we see that $\kappa > 0$. On the other hand

$$\Theta_4(g) = \frac{\int_M P v \cdot v d\mu}{\|P v\|_{L^{\frac{2n}{n-4}}}^2} = \kappa^{-1} \|v\|_{L^{\frac{2n}{n-4}}}^{-\frac{8}{n-4}}.$$

Hence

$$Y_4^+(g) = \kappa \|v\|_{L^{\frac{2n}{n-4}}}^{-\frac{8}{n-4}} = \frac{\int_M P v \cdot v d\mu}{\|v\|_{L^{\frac{2n}{n-4}}}^2}.$$

In other words, positive maximizers for $\Theta_4(g)$ are also minimizers for $Y_4^+(g)$.

2.2. The sphere S^n . On S^n ($n \geq 5$) with standard metric we have

$$Q = \frac{n(n+2)(n-2)}{8} \quad (2.7)$$

and

$$P u = \Delta^2 u - \frac{n^2 - 2n - 4}{2} \Delta u + \frac{n(n+2)(n-2)(n-4)}{16} u. \quad (2.8)$$

Let N be the north pole and $\pi_N : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the stereographic projection. Using $x = \pi_N$ as the coordinate, then the Green's function of P with pole at N is given by

$$G_{P,N} = \frac{1}{n(n-2)(n-4)2^{n-3}\omega_n} \left(|x|^2 + 1\right)^{\frac{n-4}{2}}. \quad (2.9)$$

Here ω_n is the volume of the unit ball in \mathbb{R}^n i.e.

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad (2.10)$$

Γ is the Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad \text{for } \alpha > 0. \quad (2.11)$$

From [CnLO, Li] we know

$$\begin{aligned} Y_4(S^n) &= \inf_{u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\|\Delta u\|_{L^2(\mathbb{R}^n)}^2}{\|u\|_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}^2} \\ &= \frac{\|\Delta u_1\|_{L^2(\mathbb{R}^n)}^2}{\|u_1\|_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}^2} \\ &= \frac{n(n+2)(n-2)(n-4)}{16} \frac{2^{\frac{4}{n}} \pi^{\frac{2(n+1)}{n}}}{\Gamma\left(\frac{n+1}{2}\right)^{\frac{4}{n}}} \\ &= Y_4^+(S^n). \end{aligned} \quad (2.12)$$

Here

$$u_1(x) = \left(|x|^2 + 1\right)^{-\frac{n-4}{2}}. \quad (2.13)$$

For $\lambda > 0$, let

$$u_\lambda(x) = \lambda^{-\frac{n-4}{2}} u_1\left(\frac{x}{\lambda}\right) = \left(\frac{\lambda}{|x|^2 + \lambda^2}\right)^{\frac{n-4}{2}}, \quad (2.14)$$

then

$$\Delta^2 u_\lambda = n(n+2)(n-2)(n-4) u_\lambda^{\frac{n+4}{n-4}}. \quad (2.15)$$

On the other hand it follows from [CnLO, Li] that

$$\begin{aligned}
& \Theta_4(S^n) \\
&= \frac{1}{2n(n-2)(n-4)\omega_n} \sup_{f \in L^2(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^{n-4}} dx dy}{\|f\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}^2} \\
&= \sup_{u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta u)^2 dx}{\|\Delta^2 u\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}^2} \\
&= \frac{1}{2n(n-2)(n-4)\omega_n} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f_1(x)f_1(y)}{|x-y|^{n-4}} dx dy}{\|f_1\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}^2} \\
&= \frac{1}{Y_4(S^n)}.
\end{aligned} \tag{2.16}$$

Here

$$f_1(x) = \left(|x|^2 + 1\right)^{-\frac{n+4}{2}}. \tag{2.17}$$

For $\lambda > 0$, let

$$f_\lambda(x) = \lambda^{-\frac{n+4}{2}} f_1\left(\frac{x}{\lambda}\right) = \left(\frac{\lambda}{|x|^2 + \lambda^2}\right)^{\frac{n+4}{2}}, \tag{2.18}$$

then

$$\Delta^2 u_\lambda = n(n+2)(n-2)(n-4)f_\lambda. \tag{2.19}$$

2.3. Concentration compactness principle. Here we apply the concentration compactness principle in [Ln] to extremal problem (2.2). To achieve this goal we start with an almost sharp Sobolev inequality. Recall by (2.16) for $u \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx \leq \Theta_4(S^n) \|\Delta^2 u\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}^2. \tag{2.20}$$

Lemma 2.3. *Assume M is a smooth compact Riemannian manifold with dimension $n \geq 5$. Then for any $\varepsilon > 0$, we have*

$$\|\Delta u\|_{L^2(M)}^2 \leq (\Theta_4(S^n) + \varepsilon) \|Pu\|_{L^{\frac{2n}{n+4}}(M)}^2 + C(\varepsilon) \|u\|_{L^{\frac{2n}{n+4}}(M)}^2 \tag{2.21}$$

for all $u \in W^{4, \frac{2n}{n+4}}(M)$.

The passage from (2.20) to (2.21) is standard and we refer the readers to [DHL, He] for further details. The above almost sharp Sobolev inequality can be used to prove the following concentration compactness lemma. We refer the readers to [He, Ln] for the now standard argument.

Lemma 2.4. *Let M be a smooth compact Riemannian manifold with dimension $n \geq 5$, $\ker P = 0$, $f_i \in L^{\frac{2n}{n+4}}(M)$ such that $f_i \rightharpoonup f$ weakly in $L^{\frac{2n}{n+4}}$. Let $u_i, u \in W^{4, \frac{2n}{n+4}}(M)$ such that $Pu_i = f_i, Pu = f$. Assume*

$$|f_i|^{\frac{2n}{n+4}} d\mu \rightharpoonup \sigma \text{ in } \mathcal{M}(M)$$

and

$$|\Delta u_i|^2 d\mu \rightharpoonup \nu \text{ in } \mathcal{M}(M),$$

here $\mathcal{M}(M)$ is the space of all Radon measures on M . Then there exists countably many points $p_i \in M$ such that

$$\sigma \geq |f|^{\frac{2n}{n+4}} d\mu + \sum_i \sigma_i \delta_{p_i}$$

and

$$\nu = |\Delta u|^2 d\mu + \sum_i \nu_i \delta_{p_i},$$

here $\sigma_i = \sigma(\{p_i\})$, $\nu_i = \nu(\{p_i\})$. Moreover

$$\nu_i \leq \Theta_4(S^n) \sigma_i^{\frac{n+4}{n}}.$$

Now we are ready to derive a criterion for the existence of maximizers. Such kind of criterion is an analog statement for those of Yamabe problems ([LP]).

Proposition 2.1. *Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $\ker P = 0$. Let*

$$\Theta_4(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2}.$$

If $\Theta_4(g) > \Theta_4(S^n)$ and $f_i \in L^{\frac{2n}{n+4}}$ satisfies $\|f_i\|_{L^{\frac{2n}{n+4}}} = 1$, $\int_M G_P f_i \cdot f_i d\mu \rightarrow \Theta_4(g)$, then after passing to a subsequence, we can find a $f \in L^{\frac{2n}{n+4}}$ such that $f_i \rightarrow f$ in $L^{\frac{2n}{n+4}}$. In particular, $\|f\|_{L^{\frac{2n}{n+4}}} = 1$ and $\int_M G_P f \cdot f d\mu = \Theta_4(g)$, f is a maximizer for $\Theta_4(g)$.

Proof. After passing to a subsequence we can assume $f_i \rightharpoonup f$ weakly in $L^{\frac{2n}{n+4}}$. Let $u_i, u \in W^{4, \frac{2n}{n+4}}$ such that $Pu_i = f_i$, $Pu = f$. Then $u_i \rightharpoonup u$ weakly in $W^{4, \frac{2n}{n+4}}$, $u_i \rightarrow u$ in $W^{3, \frac{2n}{n+4}}$ and $u_i \rightarrow u$ in $W^{1,2}$. After passing to another subsequence we have

$$|f_i|^{\frac{2n}{n+4}} d\mu \rightharpoonup d\sigma \text{ and } (\Delta u_i)^2 d\mu \rightharpoonup d\nu \text{ in } \mathcal{M}(M),$$

moreover it follows from Lemma 2.4 that

$$\sigma \geq |f|^{\frac{2n}{n+4}} d\mu + \sum_i \sigma_i \delta_{p_i}, \quad \nu = (\Delta u)^2 d\mu + \sum_i \nu_i \delta_{p_i},$$

here $\sigma_i = \sigma(\{p_i\})$, $\nu_i = \nu(\{p_i\})$ and

$$\nu_i \leq \Theta_4(S^n) \sigma_i^{\frac{n+4}{n}}.$$

It follows that $\sigma(M) = 1$ and

$$\begin{aligned} & \int_M G_P f_i \cdot f_i d\mu \\ &= \int_M u_i P u_i d\mu = E(u_i) \\ &= \int_M \left((\Delta u_i)^2 - 4A(\nabla u_i, \nabla u_i) + (n-2)J|\nabla u_i|^2 + \frac{n-4}{2}Q u_i^2 \right) d\mu \\ &\rightarrow E(u) + \sum_i \nu_i. \end{aligned}$$

Hence

$$\begin{aligned}
\Theta_4(g) &= E(u) + \sum_i \nu_i \\
&\leq \Theta_4(g) \|f\|_{L^{\frac{2n}{n+4}}}^2 + \Theta_4(S^n) \sum_i \sigma_i^{\frac{n+4}{n}} \\
&\leq \Theta_4(g) \left[\left(\|f\|_{L^{\frac{2n}{n+4}}} \right)^{\frac{n+4}{n}} + \sum_i \sigma_i^{\frac{n+4}{n}} \right] \\
&\leq \Theta_4(g) \left(\|f\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + \sum_i \sigma_i \right)^{\frac{n+4}{n}} \\
&\leq \Theta_4(g).
\end{aligned}$$

Hence all inequalities become equalities. In particular, $\sigma_i = 0$, $\nu_i = 0$, $\|f\|_{L^{\frac{2n}{n+4}}} = 1$.

Hence $f_i \rightarrow f$ in $L^{\frac{2n}{n+4}}$, $E(u) = \int_M G_P f \cdot f d\mu = \Theta_4(g)$. ■

2.4. Expansion of Green's function of the Paneitz operator. In [LP], the expansion formula of Green's function of conformal Laplacian operator plays an important role. Here we determine the expansion formulas for Green's function of Paneitz operator. These formulas will be crucial in the choice of test function in section 3.

We use the same strategy as [LP, section 6], but since we need to take into account lower order terms, some efforts are needed in doing the algebra. Let us introduce some notation. For $m \in \mathbb{Z}_+$, let

$$\mathcal{P}_m = \{\text{homogeneous degree } m \text{ polynomials on } \mathbb{R}^n\}, \quad (2.22)$$

and

$$\mathcal{H}_m = \{\text{harmonic degree } m \text{ homogeneous polynomials}\}. \quad (2.23)$$

Let f be a function defined on a neighborhood of 0 except at 0, namely $U \setminus \{0\}$, m be nonnegative integer, and $\theta \in \mathbb{R}$. Then we write $f = O^{(m)}(r^\theta)$ as $r \rightarrow 0$ if

$$f \in C^m(U \setminus \{0\}) \text{ and } \partial_{i_1 \dots i_k} f(x) = O(r^{\theta-k}) \text{ as } r \rightarrow 0 \quad (2.24)$$

for $k = 0, 1, \dots, m$. Here $r = |x|$.

Another useful notation is as follows. Let f be a function defined on a neighborhood of 0, namely U , m and k be nonnegative integers. Then we write $f = O_m(r^k)$ if $f \in C^m(U)$ and $f(x) = O(r^k)$ as $r \rightarrow 0$. Similarly we write $f = O_\infty(r^k)$ if $f \in C^\infty(U)$ and $f(x) = O(r^k)$ as $r \rightarrow 0$.

Let M be a smooth compact manifold with a conformal class of Riemannian metrics. For a point $p \in M$, choose a conformal normal coordinate (see [LP]) at p , x_1, \dots, x_n . Let the metric $g = g_{ij} dx_i dx_j$. Then we have

$$J(p) = 0, \quad J_i(p) = 0, \quad \Delta J(p) = -\frac{|W(p)|^2}{12(n-1)}, \quad (2.25)$$

$$A_{ij}(p) = 0, \quad A_{ijk}(p) x_i x_j x_k = 0, \quad (2.26)$$

and

$$A_{ijkl}(p) x_i x_j x_k x_l = -\frac{2}{9(n-2)} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 - \frac{r^2}{n-2} J_{ij}(p) x_i x_j. \quad (2.27)$$

Here A_{ijk} and A_{ijkl} are covariant derivatives of the Schouten tensor A (see (1.2)).

Proposition 2.2. *Assume $n \geq 5$ and $\ker P = 0$. Then in conformal normal coordinate at p , we have the following statements:*

- *If the original conformal class is conformal flat in a neighborhood of p , then we may choose g such that it is flat near p , and*

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + O_\infty(1). \quad (2.28)$$

- *If n is odd, then*

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} \left(1 + \sum_{i=4}^n \psi_i \right) + O_4(1). \quad (2.29)$$

Here $\psi_i \in \mathcal{P}_i$.

- *If n is even and larger than or equal to 8, then*

$$\begin{aligned} & 2n(2-n)(4-n)\omega_n G_{P,p} \\ &= r^{4-n} \left(1 + \sum_{i=4}^n \psi_i \right) + r^{4-n} \log r \sum_{i=n-4}^n \psi'_i + r^{4-n} \log^2 r \sum_{i=n-2}^n \psi''_i \\ & \quad + r^{4-n} \log^3 r \cdot \psi'''_n + O_4(1). \end{aligned} \quad (2.30)$$

Here $\psi_i, \psi'_i, \psi''_i, \psi'''_i \in \mathcal{P}_i$.

- *If $n = 6$, then*

$$\begin{aligned} 96\omega_6 G_{P,p} &= r^{-2} (1 + \psi_4 + \psi_5 + \psi_6) + r^{-2} \log r (\psi'_4 + \psi'_5 + \psi'_6) \\ & \quad + r^{-2} \log^2 r \cdot \psi''_6 + O_4(1). \end{aligned} \quad (2.31)$$

Here $\psi_i, \psi'_i, \psi''_i \in \mathcal{P}_i$.

As a consequence, we have

- *If $n = 5, 6, 7$ or M is conformal flat near p , then*

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + A + O^{(4)}(r). \quad (2.32)$$

Here A is a constant.

- *If $n = 8$, then*

$$384\omega_8 G_{P,p} = r^{-4} - \frac{|W(p)|^2}{1440} \log r + O^{(4)}(1). \quad (2.33)$$

- *If $n \geq 9$, then*

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + r^{4-n} \psi_4 + O^{(4)}(r^{9-n}), \quad (2.34)$$

here $\psi_4 \in \mathcal{P}_4$ and in fact

$$\begin{aligned} \psi_4 &= \frac{1}{40(n-2)} \left[\frac{2}{9} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 - \frac{2r^2}{9(n+4)} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ &\quad \left. + \frac{|W(p)|^2}{3(n+2)(n+4)} r^4 \right] + \frac{r^2}{48(n-6)} \left[\frac{4}{9(n+4)} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ &\quad \left. - 2(n-6) J_{ij}(p) x_i x_j - \frac{(n^2 + 6n - 32) |W(p)|^2}{6n(n+4)(n-1)} r^2 \right] \\ &\quad + r^4 \cdot \frac{(n-4)(3n^2 - 2n - 64) |W(p)|^2}{576n(n+2)(n-1)(n-6)(n-8)}. \end{aligned} \quad (2.35)$$

The terms in the square brackets are harmonic polynomials.

To derive these expansions, we need some algebraic preparations. Note that \mathcal{P}_m has the following decomposition (see [S])

$$\mathcal{P}_m = \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} (r^{2k} \mathcal{H}_{m-2k}). \quad (2.36)$$

Under this decomposition, we have

$$(r^2 \Delta)|_{r^{2k} \mathcal{H}_{m-2k}} = 2k(2m - 2k + n - 2) \quad \text{for } k = 0, 1, 2, \dots, \lfloor \frac{m}{2} \rfloor. \quad (2.37)$$

Here Δ denotes the Laplace operator with respect to the Euclidean metric.

For $\alpha \in \mathbb{R}$, let

$$A_\alpha = r^2 \Delta + 2\alpha r \partial_r + \alpha(\alpha + n - 2), \quad (2.38)$$

and

$$B_\alpha = \frac{\partial}{\partial \alpha} A_\alpha = 2r \partial_r + (2\alpha + n - 2), \quad (2.39)$$

then

$$\begin{aligned} \Delta(r^\alpha \varphi) &= r^{\alpha-2} A_\alpha \varphi, \\ A_\alpha(r^\beta \varphi) &= r^\beta A_{\alpha+\beta} \varphi, \\ A_\alpha(\varphi \log r) &= (A_\alpha \varphi) \log r + B_\alpha \varphi, \\ B_\alpha(r^\beta \varphi) &= r^\beta B_{\alpha+\beta} \varphi, \\ B_\alpha(\varphi \log r) &= (B_\alpha \varphi) \log r + 2\varphi. \end{aligned}$$

In addition,

$$A_\alpha|_{\mathcal{P}_m} = r^2 \Delta + \alpha(2m + \alpha + n - 2), \quad (2.40)$$

$$B_\alpha|_{\mathcal{P}_m} = 2m + 2\alpha + n - 2, \quad (2.41)$$

and

$$A_\alpha|_{r^{2k} \mathcal{H}_{m-2k}} = (\alpha + 2k)(2m - 2k + \alpha + n - 2) \quad (2.42)$$

for $k = 0, 1, 2, \dots, \lfloor \frac{m}{2} \rfloor$. In particular,

$$\begin{aligned} &(A_{2-n} A_{4-n})|_{r^{2k} \mathcal{H}_{m-2k}} \\ &= (2m - 2k)(2m - 2k + 2)(2k + 2 - n)(2k + 4 - n), \end{aligned} \quad (2.43)$$

for $k = 0, 1, 2, \dots, \lfloor \frac{m}{2} \rfloor$.

Lemma 2.5. *For any real numbers α and β , and any nonnegative integer k , we have*

$$\begin{aligned} B_\alpha \left(\varphi \log^k r \right) &= B_\alpha \varphi \cdot \log^k r + 2k \varphi \log^{k-1} r, \\ A_\alpha \left(\varphi \log^k r \right) &= A_\alpha \varphi \cdot \log^k r + k B_\alpha \varphi \cdot \log^{k-1} r + k(k-1) \varphi \log^{k-2} r, \end{aligned}$$

and

$$\begin{aligned} &A_\alpha A_\beta \left(\varphi \log^k r \right) \\ &= A_\alpha A_\beta \varphi \cdot \log^k r + k (A_\alpha B_\beta \varphi + B_\alpha A_\beta \varphi) \log^{k-1} r \\ &\quad + k(k-1) (A_\alpha \varphi + A_\beta \varphi + B_\alpha B_\beta \varphi) \log^{k-2} r \\ &\quad + k(k-1)(k-2) (B_\alpha \varphi + B_\beta \varphi) \log^{k-3} r \\ &\quad + k(k-1)(k-2)(k-3) \varphi \log^{k-4} r. \end{aligned}$$

Proof. Observe

$$\frac{\partial}{\partial \alpha} B_\alpha \varphi = 2\varphi, \quad \frac{\partial^2}{\partial \alpha^2} B_\alpha \varphi = 0.$$

Now since $B_\alpha (r^\beta \varphi) = r^\beta B_{\alpha+\beta} \varphi$, we know

$$\begin{aligned} B_\alpha \left(\varphi \log^k r \right) &= \left. \frac{\partial^k}{\partial \beta^k} \right|_{\beta=0} B_\alpha (r^\beta \varphi) \\ &= \left. \frac{\partial^k}{\partial \beta^k} \right|_{\beta=0} (r^\beta B_{\alpha+\beta} \varphi) \\ &= B_\alpha \varphi \cdot \log^k r + 2k \varphi \log^{k-1} r, \end{aligned}$$

here we have used the Newton-Lebniz formula. For the second equation, we start with

$$\frac{\partial}{\partial \alpha} A_\alpha \varphi = B_\alpha \varphi, \quad \frac{\partial^2}{\partial \alpha^2} A_\alpha \varphi = 2\varphi, \quad \frac{\partial^3}{\partial \alpha^3} A_\alpha \varphi = 0,$$

then

$$\begin{aligned} A_\alpha \left(\varphi \log^k r \right) &= \left. \frac{\partial^k}{\partial \beta^k} \right|_{\beta=0} A_\alpha (r^\beta \varphi) \\ &= \left. \frac{\partial^k}{\partial \beta^k} \right|_{\beta=0} (r^\beta A_{\alpha+\beta} \varphi) \\ &= A_\alpha \varphi \cdot \log^k r + k B_\alpha \varphi \cdot \log^{k-1} r + k(k-1) \varphi \log^{k-2} r. \end{aligned}$$

■

Define an operator

$$M_g \varphi = 4 \operatorname{div} (A (\nabla_g \varphi, e_i) e_i) + (2-n) \operatorname{div} (J \nabla_g \varphi). \quad (2.44)$$

The Paneitz operator can be written as

$$P_g \varphi = \Delta_g^2 \varphi + M_g \varphi + \frac{n-4}{2} Q \varphi. \quad (2.45)$$

For any $\alpha \in \mathbb{R}$, define

$$\begin{aligned} N_{\alpha,g}\varphi &= r^4 M_g \varphi + 8\alpha r^2 A(r\partial_r, \nabla_g \varphi) + 2(2-n)\alpha r^2 J \cdot r\partial_r \varphi \\ &\quad + 4\alpha r^2 \operatorname{div}(A(r\partial_r, e_i)e_i)\varphi + (2-n)\alpha r^2 \cdot r\partial_r J \cdot \varphi \\ &\quad + 4\alpha(\alpha-2)A(r\partial_r, r\partial_r)\varphi + (2-n)\alpha(\alpha+n-2)r^2 J\varphi, \end{aligned} \quad (2.46)$$

then

$$M_g(r^\alpha \varphi) = r^{\alpha-4} N_{\alpha,g} \varphi. \quad (2.47)$$

At first, we claim that

$$P_g(r^{4-n}) = 2n(2-n)(4-n)\omega_n \delta_p + f r^{-n}, \quad (2.48)$$

with $f = O_\infty(r^4)$.

Indeed, since r^{4-n} is radial, we have

$$\Delta_g^2(r^{4-n}) = 2n(2-n)(4-n)\omega_n \delta_p. \quad (2.49)$$

On the other hand,

$$M_g(r^{4-n}) = r^{-n} N_{4-n,g} 1.$$

In view of the facts

$$\begin{aligned} &\operatorname{div}(A(r\partial_r, e_i)e_i) \\ &= \partial_k(x_i A_{ij} g^{jk}) \\ &= g^{ij} A_{ij} + x_i \partial_k A_{ij} g^{jk} + O_\infty(r^2) \\ &= J + x_i A_{ijk}(p) \delta_{jk} + O_\infty(r^2) \\ &= x_i J_i(p) + O_\infty(r^2) \\ &= O_\infty(r^2), \end{aligned}$$

and

$$A(r\partial_r, r\partial_r) = A_{ij} x_i x_j = A_{ijk}(p) x_i x_j x_k + O_\infty(r^4) = O_\infty(r^4),$$

we see $N_{4-n,g} 1 \in O_\infty(r^4)$, (2.48) follows.

To continue, first we introduce a notation. For any $\alpha \in \mathbb{R}$, let

$$A_{\alpha,g} = r^2 \Delta_g + 2\alpha r \partial_r + \alpha(\alpha+n-2), \quad (2.50)$$

then

$$\begin{aligned} \Delta_g(r^\alpha \varphi) &= r^{\alpha-2} A_{\alpha,g} \varphi, \\ A_{\alpha,g}(r^\beta \varphi) &= r^\beta A_{\alpha+\beta,g} \varphi, \\ A_{\alpha,g}(\varphi \log r) &= A_{\alpha,g} \varphi \cdot \log r + B_\alpha \varphi. \end{aligned}$$

Note that

$$A_{\alpha,g} = A_\alpha + r^2(\Delta_g - \Delta) = A_\alpha + r^2 \partial_i((g^{ij} - \delta_{ij}) \partial_j). \quad (2.51)$$

A straightforward computation shows

$$P_g(r^\alpha \varphi) = r^{\alpha-4} (A_{\alpha-2} A_\alpha \varphi + K_\alpha \varphi), \quad (2.52)$$

where

$$\begin{aligned} &K_\alpha \varphi \\ &= A_{\alpha-2}(r^2(\Delta_g - \Delta)\varphi) + r^2(\Delta_g - \Delta)A_{\alpha,g}\varphi + N_{\alpha,g}\varphi + \frac{n-4}{2}r^4 Q\varphi. \end{aligned} \quad (2.53)$$

We easily see that for any nonnegative integer k , $\varphi = O_\infty(r^k)$ implies $K_\alpha \varphi = O_\infty(r^{k+2})$.

We also introduce the following two operators,

$$\begin{aligned} K_\alpha^{(1)} \varphi &= \frac{\partial}{\partial \alpha} K_\alpha \varphi \\ &= B_{\alpha-2} (r^2 (\Delta_g - \Delta) \varphi) + r^2 (\Delta_g - \Delta) B_\alpha \varphi \\ &\quad + 8r^2 A (r \partial_r, \nabla_g \varphi) + 2(2-n) r^2 J \cdot r \partial_r \varphi \\ &\quad + 4r^2 \operatorname{div} (A (r \partial_r, e_i) e_i) \varphi + (2-n) r^2 \cdot r \partial_r J \cdot \varphi \\ &\quad + 8(\alpha-1) A (r \partial_r, r \partial_r) \varphi + (2-n) (2\alpha+n-2) r^2 J \varphi, \end{aligned} \quad (2.54)$$

and

$$\begin{aligned} K_\alpha^{(2)} \varphi &= \frac{\partial}{\partial \alpha} K_\alpha^{(1)} \varphi \\ &= 4r^2 (\Delta_g - \Delta) \varphi + 8A (r \partial_r, r \partial_r) \varphi + 2(2-n) r^2 J \varphi \\ &= K^{(2)} \varphi. \end{aligned} \quad (2.55)$$

Clearly, $\varphi = O_\infty(r^k)$ for some nonnegative integer would imply $K_\alpha^{(1)} \varphi, K^{(2)} \varphi = O_\infty(r^{k+2})$. In addition, these operators satisfy the following

$$\begin{aligned} K_\alpha (r^\beta \varphi) &= r^\beta K_{\alpha+\beta} \varphi, \\ K_\alpha (\varphi \log r) &= K_\alpha \varphi \cdot \log r + K_\alpha^{(1)} \varphi, \\ K_\alpha^{(1)} (r^\beta \varphi) &= r^\beta K_{\alpha+\beta}^{(1)} \varphi, \\ K_\alpha^{(1)} (\varphi \log r) &= K_\alpha^{(1)} \varphi \cdot \log r + K_\alpha^{(2)} \varphi, \\ K^{(2)} (r^\beta \varphi) &= r^\beta K^{(2)} \varphi, \\ K^{(2)} (\varphi \log r) &= K^{(2)} \varphi \cdot \log r. \end{aligned}$$

More generally, we have

Lemma 2.6. *For any nonnegative integer k , we have*

$$\begin{aligned} K_\alpha^{(1)} (\varphi \log^k r) &= K_\alpha^{(1)} \varphi \cdot \log^k r + k K^{(2)} \varphi \cdot \log^{k-1} r, \\ K_\alpha (\varphi \log^k r) &= K_\alpha \varphi \cdot \log^k r + k K_\alpha^{(1)} \varphi \cdot \log^{k-1} r + \frac{k(k-1)}{2} K^{(2)} \varphi \cdot \log^{k-2} r. \end{aligned}$$

This follows from the same proof of Lemma 2.4.

Case 2.1. *The dimension n is odd.*

In this case, we claim that we may find a $\psi = \sum_{i=1}^n \psi_i$, with $\psi_i \in \mathcal{P}_i$ such that

$$A_{2-n} A_{4-n} \psi + K_{4-n} \psi + f = O_\infty(r^{n+1}). \quad (2.56)$$

Once this has been done, then we have

$$r^{-n} (A_{2-n} A_{4-n} \psi + K_{4-n} \psi + f) \in C^\alpha \quad \text{for any } 0 < \alpha < 1.$$

If the domain is small enough, then we may find $\bar{\psi} \in C^{4,\alpha}$ such that

$$P_g \bar{\psi} = -r^{-n} (A_{2-n} A_{4-n} \psi + K_{4-n} \psi + f).$$

Then

$$P_g (r^{4-n} (1 + \psi) + \bar{\psi}) = 2n(2-n)(4-n) \omega_n \delta_p. \quad (2.57)$$

Hence the Green's function satisfies

$$2n(2-n)(4-n)\omega_n G_p = r^{4-n}(1+\psi) + \overline{\psi} + O_\infty(1). \quad (2.58)$$

To define ψ_1, \dots, ψ_n , we let $\psi_1 = 0, \psi_2 = 0$ and $\psi_3 = 0$. One easily see

$$\begin{aligned} f_3 &= A_{2-n}A_{4-n}(\psi_1 + \psi_2 + \psi_3) + K_{4-n}(\psi_1 + \psi_2 + \psi_3) + f \\ &= f = O_\infty(r^4). \end{aligned} \quad (2.59)$$

Assume we have found $\psi_1, \psi_2, \dots, \psi_k$ for $3 \leq k \leq n-1$, such that $\psi_i \in \mathcal{P}_i$ and

$$f_k = A_{2-n}A_{4-n}\left(\sum_{i=1}^k \psi_i\right) + K_{4-n}\left(\sum_{i=1}^k \psi_i\right) + f = O_\infty(r^{k+1}),$$

then we write $f_k = \phi_{k+1} + O_\infty(r^{k+2}), \phi_{k+1} \in \mathcal{P}_{k+1}$. Since

$$\begin{aligned} &A_{2-n}A_{4-n}|_{r^{2j}\mathcal{H}_{k+1-2j}} \\ &= (2(k+1)-2j)(2(k+1)-2j+2)(2j+2-n)(2j+4-n) \neq 0 \end{aligned}$$

for $j = 0, 1, 2, \dots, \lfloor \frac{k+1}{2} \rfloor$, $A_{2-n}A_{4-n}$ is invertible on \mathcal{P}_{k+1} . We may find a unique $\psi_{k+1} \in \mathcal{P}_{k+1}$, such that

$$A_{2-n}A_{4-n}\psi_{k+1} + \phi_{k+1} = 0. \quad (2.60)$$

Then

$$\begin{aligned} f_{k+1} &= A_{2-n}A_{4-n}\left(\sum_{i=1}^{k+1} \psi_i\right) + K_{4-n}\left(\sum_{i=1}^{k+1} \psi_i\right) + f \\ &= f_k + A_{2-n}A_{4-n}\psi_{k+1} + K_{4-n}\psi_{k+1} = O_\infty(r^{k+2}). \end{aligned}$$

This finishes the induction process.

Case 2.2. n is even and larger than or equal to 8.

In this case, we first set $\psi_1 = 0, \psi_2 = 0$ and $\psi_3 = 0$. Since $A_{2-n}A_{4-n}$ is invertible on \mathcal{P}_k for $0 \leq k \leq n-5$, by the same induction procedure as Case 2.1, we can find $\psi_4, \dots, \psi_{n-5}$ such that $\psi_i \in \mathcal{P}_i$ and

$$f_{n-5} = A_{2-n}A_{4-n}\left(\sum_{i=1}^{n-5} \psi_i\right) + K_{4-n}\left(\sum_{i=1}^{n-5} \psi_i\right) + f = O_\infty(r^{n-4}).$$

To continue, we write

$$f_{n-5} = \phi_{n-4} + O_\infty(r^{n-3}), \quad \phi_{n-4} \in \mathcal{P}_{n-4}.$$

Let $\psi_{n-4}^{(0)} = \alpha_{n-4}^{(0)} + \beta_{n-4}^{(0)} \log r$ with $\alpha_{n-4}^{(0)}, \beta_{n-4}^{(0)} \in \mathcal{P}_{n-4}$, then

$$\begin{aligned} &A_{2-n}A_{4-n}\psi_{n-4}^{(0)} \\ &= A_{2-n}A_{4-n}\alpha_{n-4}^{(0)} + (A_{2-n}B_{4-n} + B_{2-n}A_{4-n})\beta_{n-4}^{(0)} + A_{2-n}A_{4-n}\beta_{n-4}^{(0)} \cdot \log r. \end{aligned}$$

Let $\beta_{n-4}^{(0)} \in r^{n-4}\mathcal{H}_0$, then since

$$(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-4}\mathcal{H}_0} = -2(n-2)(n-4) \neq 0,$$

and

$$\begin{aligned} &A_{2-n}A_{4-n}|_{r^{2k}\mathcal{H}_{n-4-2k}} \\ &= (2(n-4)-2k)(2(n-4)-2k+2)(2k+2-n)(2k+4-n) \neq 0, \end{aligned}$$

for $0 \leq k \leq \frac{n}{2} - 3$, we may find a $\alpha_{n-4}^{(0)} \in \mathcal{P}_{n-4}$ and a $\beta_{n-4}^{(0)} \in r^{n-4}\mathcal{H}_0$ such that

$$A_{2-n}A_{4-n}\psi_{n-4}^{(0)} + \phi_{n-4} = 0.$$

This implies

$$\begin{aligned} f_{n-4} &= A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} \right) + K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} \right) + f \\ &= f_{n-5} + A_{2-n}A_{4-n}\psi_{n-4}^{(0)} + K_{4-n}\psi_{n-4}^{(0)} \\ &= O_\infty(r^{n-3}) + O_\infty(r^{n-2}) \log r. \end{aligned}$$

Next we write

$$f_{n-4} = \phi_{n-3} + O_\infty(r^{n-2}) \log r + O_\infty(r^{n-2}), \quad \phi_{n-3} \in \mathcal{P}_{n-3}.$$

Again by similar arguments, we can find $\psi_{n-3}^{(0)} \in \mathcal{P}_{n-3} + r^{n-4}\mathcal{H}_1 \log r$ such that

$$A_{2-n}A_{4-n}\psi_{n-3}^{(0)} + \phi_{n-3} = 0.$$

Then

$$\begin{aligned} f_{n-3} &= A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} \right) + K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} \right) + f \\ &= f_{n-4} + A_{2-n}A_{4-n}\psi_{n-3}^{(0)} + K_{4-n}\psi_{n-3}^{(0)} \\ &= O_\infty(r^{n-2}) \log r + O_\infty(r^{n-2}). \end{aligned}$$

We write

$$f_{n-3} = \phi_{n-2}^{(1)} \log r + O_\infty(r^{n-2}) + O_\infty(r^{n-1}) \log r.$$

Similar as before, we may find

$$\psi_{n-2}^{(1)} \in \mathcal{P}_{n-2} \log r + (r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2) \log^2 r$$

such that

$$A_{2-n}A_{4-n}\psi_{n-2}^{(1)} + \phi_{n-2}^{(1)} \log r \in \mathcal{P}_{n-2}.$$

Indeed, for $\psi_{n-2}^{(1)} = \alpha_{n-2}^{(1)} \log r + \beta_{n-2}^{(1)} \log^2 r$, with $\alpha_{n-2}^{(1)}, \beta_{n-2}^{(1)} \in \mathcal{P}_{n-2}$, we have

$$\begin{aligned} &A_{2-n}A_{4-n}\psi_{n-2}^{(1)} \\ &= \left(A_{2-n}A_{4-n}\alpha_{n-2}^{(1)} + 2(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})\beta_{n-2}^{(1)} \right) \log r \\ &\quad + A_{2-n}A_{4-n}\beta_{n-2}^{(1)} \cdot \log^2 r + \mathcal{P}_{n-2}. \end{aligned}$$

Let $\beta_{n-2}^{(1)} \in r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2$. Since

$$\begin{aligned} 2(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-2}\mathcal{H}_0} &= 4n(n-2) \neq 0, \\ 2(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-4}\mathcal{H}_2} &= -4n(n+2) \neq 0, \end{aligned}$$

and

$$\begin{aligned} &A_{2-n}A_{4-n}|_{r^{2k}\mathcal{H}_{n-2-2k}} \\ &= (2(n-2) - 2k)(2(n-2) - 2k + 2)(2k + 2 - n)(2k + 4 - n) \neq 0 \end{aligned}$$

for $0 \leq k \leq \frac{n}{2} - 3$, we may find the above needed $\psi_{n-2}^{(1)}$. Then

$$\begin{aligned}
& f_{n-2}^{(1)} \\
&= A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} + \psi_{n-2}^{(1)} \right) \\
&\quad + K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} + \psi_{n-2}^{(1)} \right) + f \\
&= f_{n-3} + A_{2-n}A_{4-n}\psi_{n-2}^{(1)} + K_{4-n}\psi_{n-2}^{(1)} \\
&= O_\infty(r^{n-2}) + O_\infty(r^{n-1})\log r + O_\infty(r^n)\log^2 r.
\end{aligned}$$

The next step is to remove the \mathcal{P}_{n-2} term in $O_\infty(r^{n-2})$, then the $\mathcal{P}_{n-1}\log r$ term in $O_\infty(r^{n-1})\log r$ and so on, until we reach $O_\infty(r^{n+1})\log^2 r + O_\infty(r^{n+1})\log r + O_\infty(r^{n+1}) + O_\infty(r^{n+2})\log^3 r$. That is, we find

$$\begin{aligned}
\psi_{n-4}^{(0)} &\in \mathcal{P}_{n-4} + r^{n-4}\mathcal{H}_0 \log r, \\
\psi_{n-3}^{(0)} &\in \mathcal{P}_{n-3} + r^{n-4}\mathcal{H}_1 \log r, \\
\psi_{n-2}^{(1)} &\in \mathcal{P}_{n-2} \log r + (r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2) \log^2 r, \\
\psi_{n-2}^{(0)} &\in \mathcal{P}_{n-2} + (r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2) \log r, \\
\psi_{n-1}^{(1)} &\in \mathcal{P}_{n-1} \log r + (r^{n-2}\mathcal{H}_1 + r^{n-4}\mathcal{H}_3) \log^2 r, \\
\psi_{n-1}^{(0)} &\in \mathcal{P}_{n-1} + (r^{n-2}\mathcal{H}_1 + r^{n-4}\mathcal{H}_3) \log r, \\
\psi_n^{(2)} &\in \mathcal{P}_n \log^2 r + (r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4) \log^3 r, \\
\psi_n^{(1)} &\in \mathcal{P}_n \log r + (r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4) \log^2 r,
\end{aligned}$$

and

$$\psi_n^{(0)} \in \mathcal{P}_n + (r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4) \log r,$$

such that

$$\begin{aligned}
f_n &= A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)} \right) \\
&\quad + K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)} \right) + f \\
&= O_\infty(r^{n+1})\log^2 r + O_\infty(r^{n+1})\log r + O_\infty(r^{n+1}) + O_\infty(r^{n+2})\log^3 r.
\end{aligned}$$

Clearly $r^{-n}f_n \in C^\alpha$ for any $0 < \alpha < 1$. This implies locally we may find $\bar{\psi} \in C^{4,\alpha}$ such that $P_g \bar{\psi} = -r^{-n}f_n$. Let

$$\psi = \sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)},$$

then

$$P_g(r^{4-n}(1+\psi) + \bar{\psi}) = 2n(2-n)(4-n)\omega_n\delta_p$$

on a small disk. Hence

$$2n(2-n)(4-n)\omega_n G_p = r^{4-n}(1+\psi) + \bar{\psi} + O_\infty(1).$$

Case 2.3. $n = 6$.

This case can be done similarly as for Case 2.2. That is, we can find

$$\begin{aligned}\psi_4^{(0)} &\in \mathcal{P}_4 + (r^4 \mathcal{H}_0 + r^2 \mathcal{H}_2) \log r, \\ \psi_5^{(0)} &\in \mathcal{P}_5 + (r^4 \mathcal{H}_1 + r^2 \mathcal{H}_3) \log r, \\ \psi_6^{(1)} &\in \mathcal{P}_6 \log r + (r^4 \mathcal{H}_2 + r^2 \mathcal{H}_4) \log^2 r,\end{aligned}$$

and

$$\psi_6^{(0)} \in \mathcal{P}_6 + (r^4 \mathcal{H}_2 + r^2 \mathcal{H}_4) \log r,$$

such that

$$\begin{aligned}f_6 &= A_{-4} A_{-2} \left(\psi_4^{(0)} + \psi_5^{(0)} + \psi_6^{(0)} + \psi_6^{(1)} \right) + K_{-2} \left(\psi_4^{(0)} + \psi_5^{(0)} + \psi_6^{(0)} + \psi_6^{(1)} \right) + f \\ &= O_\infty(r^7) \log r + O_\infty(r^7) + O_\infty(r^8) \log^2 r.\end{aligned}$$

The remaining argument can be done as before.

Case 2.4. M is conformal flat near p .

In this case, we may take the metric g such that it is flat near p . This implies $P_g = \Delta^2$, and hence

$$P_g(r^{4-n}) = 2n(2-n)(4-n)\omega_n \delta_p.$$

It follows that

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + O_\infty(1).$$

Finally, to get the leading terms in the expansion for $n \geq 8$, by a direct computation we have $f_3 = f = \phi_4 + O_\infty(r^5)$, with $\phi_4 \in \mathcal{P}_4$ and

$$\begin{aligned}\phi_4 &= -\frac{4(n-4)}{9} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 + 2(n-4)(n-6) r^2 J_{ij}(p) x_i x_j \\ &\quad + \frac{(n-4)|W(p)|^2}{24(n-1)} r^4.\end{aligned}\tag{2.61}$$

From this, we can compute the leading terms of $G_{P,p}$ directly from the arguments in Case 2.2.

3. EXISTENCE AND REGULARITY OF MAXIMIZERS

The main aim of this section is to show the strict inequality between $\Theta_4(g)$ and $\Theta_4(S^n)$ in the assumption of Proposition 2.1 is valid as long as (M, g) is not conformally equivalent to the standard sphere. As in the Yamabe problem case ([LP]), this is achieved by a careful choice of test function.

Proposition 3.1. *Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then*

$$\Theta_4(g) \geq \Theta_4(S^n)\tag{3.1}$$

and equality holds if and only if (M, g) is conformally equivalent to the standard sphere.

Before we start the proof of Proposition 3.1, we list several basic identities which will facilitate the calculations. For $b > -n$ and $2a - b > n$,

$$\int_{\mathbb{R}^n} \frac{|x|^b}{(|x|^2 + 1)^a} dx = \frac{n\omega_n}{2} \frac{\Gamma(\frac{b+n}{2}) \Gamma(a - \frac{b+n}{2})}{\Gamma(a)} = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{b+n}{2}) \Gamma(a - \frac{b+n}{2})}{\Gamma(a) \Gamma(\frac{n}{2})}. \quad (3.2)$$

If we fix an orthonormal frame at p , and let Δ be the Euclidean Laplacian, then

$$\begin{aligned} & \Delta \sum_{k,l} (W_{ikjl}(p) x_i x_j)^2 \\ &= 4W_{ikjl}(p) W_{ikml}(p) x_j x_m + 4W_{ikjl}(p) W_{ilmk}(p) x_j x_m \\ &= 2 \sum_{ikl} (W_{ikjl}(p) x_j + W_{iljk}(p) x_j)^2 \\ &= 2 \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2, \end{aligned} \quad (3.3)$$

and

$$\Delta^2 \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 = 8 \left(|W(p)|^2 + W_{ikjl}(p) W_{iljk}(p) \right) = 12 |W(p)|^2, \quad (3.4)$$

here we have used

$$W_{ikjl}(p) W_{iljk}(p) = \frac{1}{2} |W(p)|^2, \quad (3.5)$$

which follows from the usual Bianchi identity. Hence

$$\begin{aligned} & \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 \\ &= \left[\sum_{kl} (W_{ikjl}(p) x_i x_j)^2 - \frac{r^2}{n+4} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ & \quad \left. + \frac{3}{2(n+2)(n+4)} |W(p)|^2 r^4 \right] + r^2 \cdot \left[\frac{1}{n+4} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ & \quad \left. - \frac{3}{n(n+4)} |W(p)|^2 r^2 \right] + r^4 \cdot \frac{3}{2n(n+2)} |W(p)|^2. \end{aligned} \quad (3.6)$$

The polynomials in the square brackets are harmonic. In particular,

$$\int_{S^{n-1}} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 dS = \frac{3\omega_n}{2(n+2)} |W(p)|^2. \quad (3.7)$$

Due to the fact that in (2.2) the power $\frac{2n}{n+4} < 2 < \frac{2n}{n-4}$, to control the error on annulus region, the choice of test functions for $\Theta_4(g)$ will be more delicate than those for the classical Yamabe problem or for $Y_4(g)$ (see (1.13)) in the literature. In particular, dimension 8 and 9 has to be separated from dimensions 5, 6, 7 and dimensions greater than 9.

Fix a function $\eta_1 \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\eta_1|_{(-\infty, 1)} = 0$, $\eta_1|_{(2, \infty)} = 1$ and $0 \leq \eta_1 \leq 1$. Denote $\eta_2 = 1 - \eta_1$. For convenience we always denote

$$H = 2n(2-n)(4-n)\omega_n G_{P,p}. \quad (3.8)$$

Let δ be a small positive number. For $0 < \lambda < \delta$, let

$$u_\lambda = \left(\frac{\lambda}{|x|^2 + \lambda^2} \right)^{\frac{n-4}{2}} \quad (3.9)$$

and

$$\beta = \lambda^{\frac{n-4}{2}} r^{4-n} - u_\lambda. \quad (3.10)$$

If we write $\phi(x) = |x|^{4-n} - (|x|^2 + 1)^{\frac{4-n}{2}}$, then $\beta = \lambda^{\frac{4-n}{2}} \phi\left(\frac{x}{\lambda}\right)$.

Case 3.1. M is conformally flat near p , $n \geq 5$.

In this case we can assume that the metric g is flat near p . Using the Euclidean coordinate at p , namely x_1, \dots, x_n we have

$$H = r^{4-n} + A_0 + \alpha. \quad (3.11)$$

Here A_0 is a constant, $\alpha = O_\infty(r)$ is a biharmonic function (with respect to Euclidean metric).

Define

$$\varphi_\lambda = \begin{cases} u_\lambda + \eta_1\left(\frac{r}{\delta}\right)\beta + \lambda^{\frac{n-4}{2}}A_0 + \lambda^{\frac{n-4}{2}}\alpha, & \text{on } B_{3\delta}(p), \\ \lambda^{\frac{n-4}{2}}H, & \text{on } M \setminus B_{3\delta}(p). \end{cases} \quad (3.12)$$

It is clear that $\varphi_\lambda \in C^\infty(M)$. Note that

$$\begin{aligned} & P\varphi_\lambda \\ &= \begin{cases} n(n+2)(n-2)(n-4)\lambda^{\frac{n+4}{2}}(|x|^2 + \lambda^2)^{-\frac{n+4}{2}}, & \text{on } B_\delta(p), \\ O\left(\lambda^{\frac{n}{2}}\right), & \text{on } B_{2\delta}(p) \setminus B_\delta(p), \\ 0, & \text{on } M \setminus B_{2\delta}(p). \end{cases} \end{aligned} \quad (3.13)$$

Hence

$$\begin{aligned} & \int_M |P\varphi_\lambda|^{\frac{2n}{n+4}} d\mu \\ &= (n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}} \frac{\Gamma\left(\frac{n}{2}\right) \pi^{\frac{n}{2}}}{(n-1)!} + O\left(\lambda^{\frac{n^2}{n+4}}\right). \end{aligned} \quad (3.14)$$

It follows that

$$\begin{aligned} & \|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2 \\ &= (n(n+2)(n-2)(n-4))^2 \frac{\Gamma\left(\frac{n}{2}\right)^{\frac{n+4}{n}} \pi^{\frac{n+4}{2}}}{((n-1)!)^{\frac{n+4}{n}}} + O\left(\lambda^{\frac{n^2}{n+4}}\right). \end{aligned} \quad (3.15)$$

On the other hand,

$$\begin{aligned} & \int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu \\ &= n(n+2)(n-2)(n-4) \frac{\Gamma\left(\frac{n}{2}\right) \pi^{\frac{n}{2}}}{(n-1)!} + \frac{4(n-2)(n-4) \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} A_0 \lambda^{n-4} + o(\lambda^{n-4}). \end{aligned} \quad (3.16)$$

Hence

$$\begin{aligned} & \frac{\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu}{\|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2} \\ &= \Theta_4(S^n) + \frac{4((n-1)!)^{\frac{n+4}{n}}}{n^2(n+2)^2(n-2)(n-4)\Gamma(\frac{n}{2})^{\frac{2n+4}{n}}\pi^2} A_0 \lambda^{n-4} + o(\lambda^{n-4}). \end{aligned} \quad (3.17)$$

If (M, g) is not conformally diffeomorphic to the standard sphere, then it follows from the arguments in [HY4, section 6] that $A_0 > 0$. Fix δ small and let $\lambda \downarrow 0$, we see $\Theta_4(g) > \Theta_4(S^n)$.

Case 3.2. $n = 5, 6, 7$.

In this case by a conformal change of the metric we can assume \exp_p preserves the volume near p (note this is another way of saying we choose a conformal normal coordinate, see [LP]). Using the normal coordinate at p , x_1, \dots, x_n , we have

$$H = r^{4-n} + A_0 + \alpha. \quad (3.18)$$

Here A_0 is a constant and $\alpha = O^{(4)}(r)$. Define

$$\varphi_\lambda = \begin{cases} u_\lambda + \eta_1\left(\frac{r}{\delta}\right)\beta + \lambda^{\frac{n-4}{2}}A_0 + \lambda^{\frac{n-4}{2}}\alpha, & \text{on } B_{3\delta}(p), \\ \lambda^{\frac{n-4}{2}}H, & \text{on } M \setminus B_{3\delta}(p). \end{cases} \quad (3.19)$$

then $\varphi_\lambda \in W^{4, \frac{2n}{n+4}}(M)$. On $B_\delta(p) \setminus \{p\}$,

$$\begin{aligned} & P\varphi_\lambda \\ &= Pu_\lambda - \lambda^{\frac{n-4}{2}}P(r^{4-n}) \\ &= \Delta^2 u_\lambda - 4 \operatorname{div}(A(\nabla\beta, e_i)e_i) + (n-2) \operatorname{div}(J\nabla\beta) - \frac{n-4}{2}Q\beta \\ &= n(n+2)(n-2)(n-4)\lambda^{\frac{n+4}{2}}(|x|^2 + \lambda^2)^{-\frac{n+4}{2}} + O\left(\lambda^{\frac{n}{2}}|x|^{2-n}\right). \end{aligned} \quad (3.20)$$

Here we will need to use (2.25) and (2.26). On $B_{2\delta}(p) \setminus B_\delta(p)$,

$$P\varphi_\lambda = -P\left(\eta_2\left(\frac{r}{\delta}\right)\beta\right) = O\left(\lambda^{\frac{n}{2}}\right) \quad (3.21)$$

and on $M \setminus B_{2\delta}(p)$, $P\varphi_\lambda = 0$. Hence

$$\begin{aligned} & \int_M |P\varphi_\lambda|^{\frac{2n}{n+4}} d\mu \\ &= (n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}} \frac{\Gamma(\frac{n}{2})^{\frac{n}{2}} \pi^{\frac{n}{2}}}{(n-1)!} + o(\lambda^{n-4}), \end{aligned}$$

and

$$\begin{aligned} & \|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2 \\ &= (n(n+2)(n-2)(n-4))^2 \frac{\Gamma(\frac{n}{2})^{\frac{n+4}{n}} \pi^{\frac{n+4}{2}}}{((n-1)!)^{\frac{n+4}{n}}} + o(\lambda^{n-4}). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu \\ &= n(n+2)(n-2)(n-4) \frac{\Gamma\left(\frac{n}{2}\right) \pi^{\frac{n}{2}}}{(n-1)!} + \frac{4(n-2)(n-4) \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} A_0 \lambda^{n-4} + o(\lambda^{n-4}). \end{aligned} \quad (3.22)$$

Summing up we have

$$\begin{aligned} & \frac{\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu}{\|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2} \\ &= \Theta_4(S^n) + \frac{4((n-1)!)^{\frac{n+4}{n}}}{n^2(n+2)^2(n-2)(n-4)\Gamma\left(\frac{n}{2}\right)^{\frac{2n+4}{n}}\pi^2} A_0 \lambda^{n-4} + o(\lambda^{n-4}). \end{aligned} \quad (3.23)$$

By [HY4, section 6] we know when (M, g) is not conformally diffeomorphic to the standard sphere, A_0 is strictly positive. Letting $\lambda \downarrow 0$, we get $\Theta_4(g) > \Theta_4(S^n)$ in this case.

Case 3.3. (M, g) is not locally conformally flat and $n = 8$.

In this case we can choose p such that $W(p) \neq 0$. By a conformal change of the metric we can assume \exp_p preserves the volume near p . Using the normal coordinate at p , x_1, \dots, x_8 , we have

$$H = r^{-4} - \frac{|W(p)|^2}{1440} \log r + \alpha. \quad (3.24)$$

Here $\alpha = O^{(4)}(1)$. Define

$$\varphi_\lambda = \begin{cases} u_\lambda + \eta_1\left(\frac{r}{\delta}\right)\beta - \frac{|W(p)|^2}{1440}\lambda^2 \log r + \lambda^2 \alpha, & \text{on } B_{3\delta}(p), \\ \lambda^2 H, & \text{on } M \setminus B_{3\delta}(p). \end{cases} \quad (3.25)$$

Then $\varphi \in W^{4, \frac{4}{3}}(M)$. On $B_\delta(p) \setminus \{p\}$,

$$\begin{aligned} P\varphi_\lambda &= Pu_\lambda - \lambda^2 P(r^{-4}) \\ &= 1920\lambda^6 \left(|x|^2 + \lambda^2\right)^{-6} - 4 \operatorname{div}(A(\nabla\beta, e_i)e_i) + 6 \operatorname{div}(J\nabla\beta) - 2Q\beta \\ &= 1920\lambda^6 \left(|x|^2 + \lambda^2\right)^{-6} + O(\beta) + O(\beta'r) + O(\beta''r^2). \end{aligned} \quad (3.26)$$

Here we have used (2.25) and (2.26). On $B_{2\delta}(p) \setminus B_\delta(p)$,

$$P\varphi_\lambda = -P\left(\eta_2\left(\frac{r}{\delta}\right)\beta\right) = O(\lambda^4) \quad (3.27)$$

and on $M \setminus B_{2\delta}(p)$, $P\varphi_\lambda = 0$. Note that

$$\begin{aligned} \beta &= \lambda^2 r^{-4} - \lambda^2 (r^2 + \lambda^2)^{-2}, \\ \beta' &= -4\lambda^2 r^{-5} + 4\lambda^2 (r^2 + \lambda^2)^{-3} r, \\ \beta'' &= 20\lambda^2 r^{-6} - 24\lambda^2 (r^2 + \lambda^2)^{-4} r^2 + 4\lambda^2 (r^2 + \lambda^2)^{-3}. \end{aligned}$$

Hence we have

$$\int_M |P\varphi_\lambda|^{\frac{4}{3}} d\mu = \frac{1920^{\frac{4}{3}} \pi^4}{840} + O(\lambda^4), \quad (3.28)$$

and

$$\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu = \frac{1920\pi^4}{840} + \frac{\pi^4 |W(p)|^2}{90} \lambda^4 \log \frac{1}{\lambda} + O(\lambda^4).$$

It follows that

$$\frac{\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu}{\|P\varphi_\lambda\|_{L^{\frac{4}{3}}}} = \Theta_4(S^8) + \frac{210^{\frac{3}{2}}}{41472000\pi^2} |W(p)|^2 \lambda^4 \log \frac{1}{\lambda} + O(\lambda^4).$$

Hence $\Theta_4(g) > \Theta_4(S^8)$.

Case 3.4. M is not conformally flat and $n = 9$.

In this case we can choose p such that $W(p) \neq 0$. By a conformal change of metric we can assume \exp_p preserves the volume near p . Using the normal coordinate at p , x_1, \dots, x_9 , we have

$$H = r^{-5} + r^{-5}\psi_4 + \alpha. \quad (3.29)$$

Here $\alpha = O^{(4)}(1)$ and

$$\begin{aligned} & \psi_4 \quad (3.30) \\ = & \frac{1}{280} \left[\frac{2}{9} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 - \frac{2}{117} r^2 \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ & \left. + \frac{|W(p)|^2}{429} r^4 \right] + \frac{r^2}{144} \left[\frac{4}{117} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 \right. \\ & \left. - 6J_{ij}(p) x_i x_j - \frac{103}{5616} |W(p)|^2 r^2 \right] + \frac{805}{1368576} |W(p)|^2 r^4. \end{aligned}$$

Define

$$\varphi_\lambda = \begin{cases} u_\lambda + \eta_1 \left(\frac{r}{\delta} \right) \beta + \lambda^{\frac{5}{2}} r^{-5} \psi_4 + \lambda^{\frac{5}{2}} \alpha, & \text{on } B_{3\delta}(p), \\ \lambda^{\frac{5}{2}} H, & \text{on } M \setminus B_{3\delta}(p). \end{cases} \quad (3.31)$$

Then $\varphi \in W^{4, \frac{18}{13}}(M)$. On $B_\delta(p) \setminus \{p\}$,

$$\begin{aligned} P\varphi_\lambda &= Pu_\lambda - \lambda^{\frac{5}{2}} P(r^{-5}) \quad (3.32) \\ &= 3465\lambda^{\frac{13}{2}} \left(|x|^2 + \lambda^2 \right)^{-\frac{13}{2}} - 4 \operatorname{div} (A(\nabla\beta, e_i) e_i) + 7 \operatorname{div} (J\nabla\beta) - \frac{5}{2} Q\beta \\ &= 3465\lambda^{\frac{13}{2}} \left(|x|^2 + \lambda^2 \right)^{-\frac{13}{2}} - 2 \left(\frac{\beta'}{r} \right)' \frac{A_{ijkl}(p) x_i x_j x_k x_l}{r} \\ &\quad + \frac{7}{2} \left(\frac{\beta'}{r} \right)' r J_{ij}(p) x_i x_j + \frac{65}{2} \frac{\beta'}{r} J_{ij}(p) x_i x_j - \frac{5}{192} |W(p)|^2 \beta \\ &\quad + O(\beta r) + O(\beta' r^2) + O(\beta'' r^3). \end{aligned}$$

On $B_{2\delta}(p) \setminus B_\delta(p)$,

$$P\varphi_\lambda = -P \left(\eta_2 \left(\frac{r}{\delta} \right) \beta \right) = O \left(\lambda^{\frac{9}{2}} \right) \quad (3.33)$$

and on $M \setminus B_{2\delta}(p)$, $P\varphi_\lambda = 0$. Note that

$$\begin{aligned}\beta &= \lambda^{\frac{5}{2}} r^{-5} - \lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{5}{2}}, \\ \beta' &= -5\lambda^{\frac{5}{2}} r^{-6} + 5\lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{7}{2}} r, \\ \frac{\beta'}{r} &= -5\lambda^{\frac{5}{2}} r^{-7} + 5\lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{7}{2}}, \\ \beta'' &= 30\lambda^{\frac{5}{2}} r^{-7} - 35\lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{9}{2}} r^2 + 5\lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{7}{2}}, \\ \left(\frac{\beta'}{r}\right)' &= 35\lambda^{\frac{5}{2}} r^{-8} - 35\lambda^{\frac{5}{2}} (r^2 + \lambda^2)^{-\frac{9}{2}} r.\end{aligned}$$

A straightforward calculation shows

$$\begin{aligned}& \int_M |P\varphi_\lambda|^{\frac{18}{13}} d\mu \\ &= \frac{3465^{\frac{18}{13}} \pi^5}{6144} \left[1 + \left(\frac{94208}{4459455} \frac{1}{\pi} - \frac{41}{9009} \right) |W(p)|^2 \lambda^4 + o(\lambda^4) \right],\end{aligned}\tag{3.34}$$

hence

$$\begin{aligned}& \|P\varphi_\lambda\|_{L^{\frac{18}{13}}}^2 \\ &= \frac{3465^2 \pi^{\frac{65}{9}}}{6144^{\frac{18}{9}}} \left[1 + \left(\frac{94208}{3087315} \frac{1}{\pi} - \frac{41}{6237} \right) |W(p)|^2 \lambda^4 + o(\lambda^4) \right].\end{aligned}\tag{3.35}$$

On the other hand,

$$\begin{aligned}& \int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu \\ &= \frac{1155}{2048} \pi^5 \left[1 + \left(\frac{94208}{3087315} \frac{1}{\pi} - \frac{41}{12474} \right) |W(p)|^2 \lambda^4 + o(\lambda^4) \right].\end{aligned}\tag{3.36}$$

Summing up we get

$$\frac{\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu}{\|P\varphi_\lambda\|_{L^{\frac{18}{13}}}^2} = \Theta_4(S^9) \left(1 + \frac{41}{12474} |W(p)|^2 \lambda^4 + o(\lambda^4) \right).\tag{3.37}$$

Hence we see that $\Theta_4(g) > \Theta_4(S^9)$.

Case 3.5. M is not conformally flat and $n \geq 10$.

We can find a point p such that $W(p) \neq 0$. Let x_1, \dots, x_n be conformal normal coordinate at p , δ be a small fixed positive number, and

$$\varphi_\lambda = u_\lambda(x) \eta_2 \left(\frac{|x|}{\delta} \right).\tag{3.38}$$

Then on $B_{2\delta}(p) \setminus B_\delta(p)$,

$$P\varphi_\lambda = O\left(\lambda^{\frac{n-4}{2}}\right).\tag{3.39}$$

On $B_\delta(p)$,

$$\begin{aligned}
& P\varphi_\lambda \\
= & n(n+2)(n-2)(n-4)\lambda^{\frac{n+4}{2}}\left(|x|^2+\lambda^2\right)^{-\frac{n+4}{2}} \\
& -\frac{4}{9}(n-4)\lambda^{\frac{n-4}{2}}\left(|x|^2+\lambda^2\right)^{-\frac{n}{2}}\sum_{kl}(W_{ikjl}(p)x_ix_j)^2 \\
& +\frac{n-4}{2}\lambda^{\frac{n-4}{2}}\left(|x|^2+\lambda^2\right)^{-\frac{n}{2}}\left(4(n-6)|x|^2+(n^2-16)\lambda^2\right)J_{ij}(p)x_ix_j \\
& +\frac{n-4}{24(n-1)}\lambda^{\frac{n-4}{2}}|W(p)|^2\left(|x|^2+\lambda^2\right)^{-\frac{n-4}{2}} \\
& +O\left(\lambda^{\frac{n-4}{2}}\left(|x|^2+\lambda^2\right)^{-\frac{n-4}{2}}|x|\right).
\end{aligned} \tag{3.40}$$

Using the basic inequality

$$\left|1+t\left|t\right|^{\frac{2n}{n+4}}-1-\frac{2n}{n+4}t\right|\leq C\left|t\right|^{\frac{2n}{n+4}} \tag{3.41}$$

we see on $B_\delta(p)$,

$$\begin{aligned}
& |P\varphi_\lambda|^{\frac{2n}{n+4}} \\
= & (n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}}\lambda^n\left(|x|^2+\lambda^2\right)^{-n}. \\
& \left[1-\frac{8}{9}\frac{\lambda^{-4}\left(|x|^2+\lambda^2\right)^2}{(n+2)(n+4)(n-2)}\sum_{kl}(W_{ikjl}(p)x_ix_j)^2\right. \\
& +\frac{\lambda^{-4}\left(|x|^2+\lambda^2\right)^2}{(n+2)(n+4)(n-2)}\left(4(n-6)|x|^2+(n^2-16)\lambda^2\right)J_{ij}(p)x_ix_j \\
& +\frac{\lambda^{-4}|W(p)|^2}{12(n+2)(n+4)(n-1)(n-2)}\left(|x|^2+\lambda^2\right)^4 \\
& +O\left(\lambda^{-4}\left(|x|^2+\lambda^2\right)^4|x|\right)+O\left(\lambda^{-\frac{8n}{n+4}}\left(|x|^2+\lambda^2\right)^{\frac{8n}{n+4}}\right) \\
& \left.+O\left(\lambda^{-\frac{8n}{n+4}}\left(|x|^2+\lambda^2\right)^{\frac{8n}{n+4}}|x|^{\frac{2n}{n+4}}\right)\right].
\end{aligned}$$

A straightforward calculation shows

$$\begin{aligned}
& \int_M |P\varphi_\lambda|^{\frac{2n}{n+4}} d\mu \\
= & (n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}}\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{(n-1)!}. \\
& \left(1-\frac{1}{3}\frac{n^2-4n-4}{(n+2)(n+4)(n-2)(n-6)(n-8)}|W(p)|^2\lambda^4+o(\lambda^4)\right).
\end{aligned} \tag{3.42}$$

Hence

$$\begin{aligned}
& \|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2 \\
&= (n(n+2)(n-2)(n-4))^2 \frac{\pi^{\frac{n+4}{2}} \Gamma\left(\frac{n}{2}\right)^{\frac{n+4}{n}}}{((n-1)!)^{\frac{n+4}{n}}} \cdot \\
&\quad \left(1 - \frac{1}{3} \frac{n^2 - 4n - 4}{n(n+2)(n-2)(n-6)(n-8)} |W(p)|^2 \lambda^4 + o(\lambda^4)\right).
\end{aligned} \tag{3.43}$$

On the other hand,

$$\begin{aligned}
& \int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu \\
&= n(n+2)(n-2)(n-4) \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{(n-1)!} \cdot \\
&\quad \left(1 - \frac{n^2 - 4n - 4}{6n(n+2)(n-2)(n-6)(n-8)} |W(p)|^2 \lambda^4 + o(\lambda^4)\right).
\end{aligned} \tag{3.44}$$

Summing up we get

$$\begin{aligned}
& \frac{\int_M P\varphi_\lambda \cdot \varphi_\lambda d\mu}{\|P\varphi_\lambda\|_{L^{\frac{2n}{n+4}}}^2} \\
&= \Theta_4(S^n) \left(1 + \frac{n^2 - 4n - 4}{6n(n+2)(n-2)(n-6)(n-8)} |W(p)|^2 \lambda^4 + o(\lambda^4)\right).
\end{aligned} \tag{3.45}$$

It follows that $\Theta_4(g) > \Theta_4(S^n)$.

Next we turn to the regularity issue for maximizers of $\Theta_4(g)$ in (1.16). Assume $f \in L^{\frac{2n}{n+4}}(M)$, $f \geq 0$ and not identically zero, and it is a maximizer for $\Theta_4(g)$, then after scaling we have

$$G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}. \tag{3.46}$$

Note that this equation is critical in the sense that if we start with $f \in L^{\frac{2n}{n+4}}$ and use the equation, the usual bootstrap method simply ends with $f \in L^{\frac{2n}{n+4}}$ again. Approaches in deriving further regularity for such kind of equations has been well understood (see for example [DHL, ER, R, V] and so on). Here is a regularity result particularly tailored for our purpose, we refer the readers to [DHL, ER, R, V] for detailed proofs.

Lemma 3.1. *Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, $f \in L^{\frac{2n}{n+4}}(M)$, $f \geq 0$ and not identically zero, and it satisfies (3.46), then $f \in C^\infty(M)$, $f > 0$.*

Now we have all the ingredients to prove Theorem 1.3. Theorem 1.1 clearly follows from Theorem 1.3.

Proof of Theorem 1.3. If (M, g) is conformal equivalent to the standard sphere, then everything follows from discussions in Section 2.2. From now on we assume that (M, g) is not conformally equivalent to the standard sphere. By Proposition 3.1 we know that $\Theta_4(g) > \Theta_4(S^n)$. [HY4, Proposition 1.1] tells us $\ker P = 0$ and

$G_P > 0$. By Proposition 2.1 we know the set

$$\mathcal{M} = \left\{ f \in L^{\frac{2n}{n+4}}(M) : \|f\|_{L^{\frac{2n}{n+4}}(M)} = 1, \int_M G_P f \cdot f d\mu = \Theta_4(g) \right\}$$

is nonempty and compact in $L^{\frac{2n}{n+4}}(M)$. If $f \in \mathcal{M}$, we can assume $f^+ \neq 0$, then f^- must be equal to zero. Indeed

$$\begin{aligned} & \Theta_4(g) \\ &= \int_M G_P f \cdot f d\mu \\ &= \int_M (G_P f^+ \cdot f^+ - 2G_P f^+ \cdot f^- + G_P f^- \cdot f^-) d\mu \\ &\leq \int_M G_P |f| \cdot |f| d\mu \\ &\leq \Theta_4(g). \end{aligned}$$

Hence $\int_M G_P f^+ \cdot f^- d\mu = 0$. Using the fact that $G_P > 0$ and $f^+ \neq 0$, we see $f^- = 0$. In another word, f does not change sign. It follows from Lemma 3.1 that $f \in C^\infty(M)$ and $f > 0$. Moreover the compactness of \mathcal{M} under $C^\infty(M)$ topology follows from its compactness in $L^{\frac{2n}{n+4}}(M)$ and the proofs of Lemma 3.1 in [DHL, ER, R, V]. ■

4. SOME DISCUSSIONS

Here we turn to the variational problem (1.13).

Proposition 4.1. *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Q \geq 0$ and not identically zero, then*

- (1) $Y_4(g) \leq Y_4(S^n)$, here S^n has the standard metric. $Y_4(g) = Y_4(S^n)$ if and only if (M, g) is conformally diffeomorphic to the standard sphere.
- (2) $Y_4(g)$ is always achieved. Let

$$\mathcal{M}_P = \left\{ u \in H^2(M) : \|u\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_4(g) \right\}, \quad (4.1)$$

then \mathcal{M}_P is not empty. For any $\alpha \in (0, 1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and when (M, g) is not conformally diffeomorphic to the standard sphere, \mathcal{M}_P is compact under $C^{4,\alpha}$ topology.

We start with the following standard fact (see [DHL, He]).

Lemma 4.1. *Let*

$$\mathcal{M}_P = \left\{ u \in H^2(M) : \|u\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_4(g) \right\}.$$

If $Y_4(g) < Y_4(S^n)$, then \mathcal{M}_P is nonempty. Moreover for any $\alpha \in (0, 1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and it is compact in $C^{4,\alpha}$ topology.

Proof of Proposition 4.1. If (M, g) is conformal equivalent to the standard sphere, then the conclusion follows from discussions in Section 2.2. Assume (M, g) is not conformal equivalent to the standard sphere, then it follows from Lemma 2.2 and Proposition 3.1 that

$$Y_4(g) \leq \frac{1}{\Theta_4(g)} < \frac{1}{\Theta_4(S^n)} = Y_4(S^n).$$

Here we want to point out that the fact $Y_4(g) < Y_4(S^n)$ can be verified, with the help of positive mass theorem for Paneitz operator ([HuR, GM, HY4]), by choosing a particular test function in (1.13) (see [ER, R, GM]). In fact the corresponding calculation is easier than what we have in the proof of Proposition 3.1, but the statement in Proposition 3.1 is stronger. By Lemma 4.1, we know \mathcal{M}_P is nonempty and $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and it is compact in $C^{4,\alpha}(M)$ for any $\alpha \in (0, 1)$. ■

Proposition 4.2. *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $Y(g) > 0$, $Y_4(g) > 0$, $Q \geq 0$ and not identically zero. Denote*

$$\mathcal{M}_P = \left\{ u \in H^2(M) : \|u\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_4(g) \right\}$$

and

$$\mathcal{M}_\Theta = \left\{ u \in W^{4, \frac{2n}{n+4}}(M) : \|u\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } \frac{E(u)}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} = \Theta_4(g) \right\}.$$

then

- (1) $\mathcal{M}_P \subset C^\infty(M)$ and for any $u \in \mathcal{M}_P$, either $u > 0$ or $-u > 0$.
- (2) $Y_4(g) \Theta_4(g) = 1$.
- (3) $\mathcal{M}_P = \mathcal{M}_\Theta$.

Proof. By Proposition 4.1 we know \mathcal{M}_P is nonempty and for any $\alpha \in (0, 1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$. By [HY4, Proposition 1.1] we know $G_P > 0$. Assume $u \in \mathcal{M}_P$, without losing of generality we can assume $u^+ \neq 0$. Now we will use an observation in [R] to show $u > 0$. In fact u satisfies $\|u\|_{L^{\frac{2n}{n-4}}} = 1$ and

$$Pu = Y_4(g) |u|^{\frac{8}{n-4}} u.$$

Let $v = G_P(|Pu|)$, then $v \in C^{4,\alpha}(M)$, $v > 0$ and $|u| \leq v$. We have

$$Y_4(g) \leq \frac{E(v)}{\|v\|_{L^{\frac{2n}{n-4}}}^2} = Y_4(g) \frac{\int_M |u|^{\frac{n+4}{n-4}} v d\mu}{\|v\|_{L^{\frac{2n}{n-4}}}^2} \leq Y_4(g) \|v\|_{L^{\frac{2n}{n-4}}}^{-1} \leq Y_4(g).$$

Hence all the inequalities become equalities. In particular $\|v\|_{L^{\frac{2n}{n-4}}} = 1 = \|u\|_{L^{\frac{2n}{n-4}}}$. Since $v \geq |u|$, we see $v = |u|$. This together with $u^+ \neq 0$ implies $u = v > 0$. Standard bootstrap method shows $u \in C^\infty(M)$. Hence $\mathcal{M}_P \subset C^\infty(M)$, moreover when (M, g) is not conformally diffeomorphic to the standard sphere, \mathcal{M}_P is compact in $C^\infty(M)$.

For $u \in \mathcal{M}_P$, we can assume $u > 0$, then $\|u\|_{L^{\frac{2n}{n-4}}} = 1$ and

$$Pu = Y_4(g) u^{\frac{n+4}{n-4}}.$$

It follows that from this equation and Lemma 2.2 that

$$\Theta_4(g) \geq \frac{E(u)}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} = \frac{1}{Y_4(g)} \geq \Theta_4(g).$$

Hence $Y_4(g) \Theta_4(g) = 1$ and $u \in \mathcal{M}_\Theta$.

On the other hand, if $u \in \mathcal{M}_\Theta$, let $f = Pu$, then

$$\Theta_4(g) = \frac{\int_M Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} = \frac{\int_M G_P f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^2}.$$

Hence it follows from Theorem 1.3 that $f \in C^\infty(M)$ and either $f > 0$ or $-f > 0$. Without losing of generality we assume $f > 0$, then $u = G_P f \in C^\infty(M)$, $u > 0$ and

$$Pu = \kappa u^{\frac{n+4}{n-4}}$$

for some positive constant κ . Using $\|u\|_{L^{\frac{2n}{n-4}}} = 1$ we see

$$\Theta_4(g) = \frac{E(u)}{\|Pu\|_{L^{\frac{2n}{n+4}}}^2} = \frac{1}{\kappa},$$

and hence $\kappa = Y_4(g)$. It follows that $E(u) = Y_4(g)$ and hence $u \in \mathcal{M}_P$. Summing up we see $\mathcal{M}_P = \mathcal{M}_\Theta$. ■

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. It is clear Theorem 1.2 follows from Proposition 4.1 and 4.2. The compactness of \mathcal{M}_P in C^∞ topology was shown in the proof of Proposition 4.2. ■

At last we will show the approach to the Q curvature equation in Theorem 1.3 gives another way to find constant scalar curvature metric in a conformal class with positive Yamabe invariant. Here we always assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 3$ and $Y(g) > 0$. To find a conformal metric with scalar curvature 1 is the same as solving

$$L\rho = \rho^{\frac{n+2}{n-2}}, \quad \rho \in C^\infty(M), \rho > 0. \quad (4.2)$$

Here L is the conformal Laplacian operator. For any $u \in C^\infty(M)$ we write

$$\begin{aligned} E_2(u) &= \int_M Lu \cdot u d\mu \\ &= \int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \right) d\mu. \end{aligned} \quad (4.3)$$

Clearly $E_2(u)$ extends continuously to $u \in H^1(M)$. To solve (4.2), people consider the variational problem (see [LP])

$$Y(g) = \inf_{u \in H^1(M) \setminus \{0\}} \frac{E_2(u)}{\|u\|_{L^{\frac{2n}{n-2}}}^2}. \quad (4.4)$$

Denote

$$\mathcal{M}_L = \left\{ u \in H^1(M) : \|u\|_{L^{\frac{2n}{n-2}}} = 1 \text{ and } E_2(u) = Y(g) \right\}, \quad (4.5)$$

then it is well known that \mathcal{M}_L is always nonempty, $\mathcal{M}_L \subset C^\infty(M)$ and for any $u \in \mathcal{M}_L$, either $u > 0$ or $-u > 0$. If $u > 0$, then after scaling u solves (4.2). Moreover when (M, g) is not conformally diffeomorphic to the standard sphere, we have $Y(g) < Y(S^n)$ and \mathcal{M}_L is compact in C^∞ topology (see [LP, S]).

Now we turn to another approach to solve (4.2). Since $Y(g) > 0$, we know the Green's function of L exists and it is always positive. We can define an operator

$$(G_L f)(p) = \int_M G_L(p, q) f(q) d\mu(q). \quad (4.6)$$

Let $f = \rho^{\frac{n+2}{n-2}}$, then (4.2) becomes

$$G_L f = f^{\frac{n+2}{n-2}}, \quad f \in C^\infty(M), f > 0. \quad (4.7)$$

Let

$$\Theta_2(g) = \sup_{f \in L^{\frac{2n}{n+2}}(M) \setminus \{0\}} \frac{\int_M G_L f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+2}}}^2} = \sup_{u \in W^{2, \frac{2n}{n+2}}(M) \setminus \{0\}} \frac{\int_M Lu \cdot u d\mu}{\|Lu\|_{L^{\frac{2n}{n+2}}}^2}. \quad (4.8)$$

Note that this functional is considered in [DoZ]. Same argument as in the proof of Lemma 2.1 shows

$$\Theta_2(g) = \sup_{\tilde{g} \in [g]} \frac{\int_M \tilde{R} d\tilde{\mu}}{\|\tilde{R}\|_{L^{\frac{2n}{n+2}}(M, d\tilde{\mu})}^2}. \quad (4.9)$$

With the solution to Yamabe problem ([LP, S]) we can deduce

Lemma 4.2. *Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \geq 3$, $Y(g) > 0$. Denote*

$$\mathcal{M}_{\Theta_2} = \left\{ u \in W^{2, \frac{2n}{n+2}}(M) : \|u\|_{L^{\frac{2n}{n+2}}(M)} = 1 \text{ and } \frac{E_2(u)}{\|Lu\|_{L^{\frac{2n}{n+2}}}^2} = \Theta_2(g) \right\}.$$

Then

- (1) $Y(g) \Theta_2(g) = 1$.
- (2) $\mathcal{M}_L = \mathcal{M}_{\Theta_2}$.

Since the proof is essentially the same as the one for Proposition 4.2, we omit it here. Roughly speaking Lemma 4.2 tells us the maximization problem for $\Theta_2(g)$ will not produce new constant scalar curvature metrics other than those by minimizing problem for $Y(g)$. However, *without using the solution to Yamabe problem*, we can use the same argument as for Theorem 1.3 to show $\Theta_2(g) \geq \Theta_2(S^n)$, with equality holds if and only if (M, g) is conformally diffeomorphic to the standard sphere (here one needs to use the positive mass theorem); \mathcal{M}_{Θ_2} is always nonempty, $\mathcal{M}_{\Theta_2} \subset C^\infty(M)$ and any $u \in \mathcal{M}_{\Theta_2}$ must be either positive or negative; \mathcal{M}_{Θ_2} is compact in $C^\infty(M)$ when (M, g) is not conformally diffeomorphic to the standard sphere. In particular, this gives another way to solve (4.2). The details are left to interested readers.

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